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What, in My Opinion, David Ruelle Should Do in the Coming Years?

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A big part of my scientific life was in parallel to the one of D. Ruelle but actually we never worked together and even didn't speak too much. However, I always felt safe if David did some research in an area close to mine.

Certainly, I have no right and no intention to prescribe to David what to do in the coming years. There is only one way to understand properly the title of this short note. Several times in his scientific career D. Ruelle was very successful providing new general concepts and giving a deep insight into old problems. Let me mention on this occasion a few examples: the notion of Gibbs state in terms of DLR-condition (DLR stands for Dobrushin, Lanford, Ruelle), the concept of strange attractor in the work of Ruelle and Takens "On the nature of turbulence," Ruelle resonances for correlation functions. This list would be certainly incomplete without mentioning SRB-measures where the letter R stands for Ruelle.

The purpose of this text is to describe two problems in which the participation of D. Ruelle can be, in my opinion, very important.

1. BLOW-UPS OR NO BLOW-UPS IN THE 3D-NAVIER–STOKES SYSTEM

The Navier–Stokes system and other equations in hydrodynamics are typical examples of non-linear infinite-dimensional dynamical systems. It

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is hard to find a better case which shows the weakness of this part of mathematics.

The basic question has a very simple formulation: do there exist strong solutions of the three-dimensional Navier–Stokes system (SNS) and are they unique for natural initial conditions.

A negative answer could be connected with solutions which develop singularities in finite time like a tornado-type solution where infinite vorticity appears at some particular points in time and space. The usual phenomenological theory takes into account heat processes, boundary conditions, etc. but it cannot be excluded that the Navier–Stokes system itself has solutions of this type.

To be more concrete, consider 3D-Navier–Stokes system with periodic boundary conditions. Write Fourier series for the unknown velocity vector $u(x, t)$:

$$u(x, t) = \sum_{k \in \mathbb{Z}^3} u_k(t)\, e^{2\pi i(k, x)}.$$

Incompressibility condition implies $u_k \perp k$ for $k \in \mathbb{Z}^3$. The system of equations for $u_k(t)$ which is equivalent to the NSS has the form:

$$\frac{du_k}{dt} = 2\pi i \sum_{k_1} (u_{k_1}, k)\, \Pi_k u_{k-k_1} - 4\pi^2 \nu k^2 u_k + f_k. \tag{1}$$

In the right-hand side Π_k is the projection to the plane orthogonal to k, $\nu > 0$ is the viscosity and $f_k \perp k$ are Fourier components of the external force.

Formal solutions of (1) satisfy the energy estimate:

$$H(t) = \sum |u_k|^2 \leqslant \text{const}$$

provided that $\sum |f_k|^2 < \infty$. Apparently in the three-dimensional case there are no universal estimates of the enstrophy

$$E(t) = \sum |k|^2\, |u_k|^2.$$

If $f_k \equiv 0$ nothing prevents having solutions of NSS for which $H(t) \to 0$, $E(t) \to \infty$ as $t \to t_0$ where t_0 is a moment of blow up.

The first general mathematical results giving the existence of weak solutions for the system (1) were proven by J. Leray about seventy years ago (see ref. 1). Then there was a great breakthrough which started with the works of Ladyzenskaya (see ref. 2) and followed later with the works by Foias and Temam, Yudovich and others (see ref. 3) where the basic questions were answered in a positive way for general two-dimensional domains.

In the three-dimensional situation one can mention only the work by Caffarelli, Kohn, and Nirenberg (see ref. 4) giving an estimate of the Hausdorff dimension of a possible set of singularities.

The main difficulty lies in the absence of information on the role of the non-linear term in (1) and on the process of spreading of initial values in time. Even a formalization of related concepts could be very important. In our recent joint paper with Dinaburg, (see ref. 5) we made a suggestion of this type but it is not clear how far it can go.

2. MATHEMATICS OF RENORMALIZATION GROUP THEORY

The ideas of Renormalization Group Theory (RGT) penetrated into mathematics from the theory of phase transitions developed in the works by M.E. Fisher, L. Kadanoff, and K. Wilson. M. Feigenbaum used the ideas of RGT in his theory of sequences of period-doubling bifurcations. Nowadays, one can say that RGT is also a part of mathematics. Let me mention a few topics where it can play a major role in future breakthroughs.

1. Mathematical theory of critical point in statistical physics and connections with conformal field theory. A recent paper by Pinson and Spencer (see ref. 6) is an example of the progress which can be achieved in this way.

2. RGT as an extension and possibly a replacement of KAM-theory. Recently new results were obtained here by A. Davie.[2]

3. Universality of statistics on small scales in ensembles of random matrices without any special symmetry. Universality of limiting distributions in probability theory is, in general, a RGT-statement. E. Brézin and A. Zee (see ref. 7) proposed some ideas of RGT to study ensembles of random matrices but the main problems still remain open.

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[2] Private communication by K. Khanin

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